ON THE CONVERGENCE AND CHARACTER OF THE CONTINUED FRACTION

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots *$$

BY

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The principal object of the present paper is to give two theorems (nos. I and IV) concerning the character of the function represented by the continued fraction

$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots$$

Incidentally a criterion for the convergence of

$$\frac{b_1}{1} + \frac{b_2}{1} + \frac{b_3}{1} \cdots$$

is obtained, which appears to be more general than any hitherto discovered except for the special case in which the elements b_n are positive. This is stated in theorem III.

§ 1.

The first part of the paper relates to the following theorem:

THEOREM I. *If k denotes the greatest modulus of a point of condensation of the coefficients a_n of the continued fraction*

(1)
$$\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots,$$

the continued fraction will represent an analytic function within a circle of radius $1/4k$, described about the origin of the z-plane as center, and the only singularities of the function contained within the circle will be poles.

If, furthermore, any circle of smaller radius is drawn about the origin as center and from this circle each of the poles is excluded by drawing around it a small but arbitrary contour, then within the region which remains the continued fraction will converge uniformly to the analytic function as its limit.

A special case of this theorem is that in which $|a_n|$ diminishes indefinitely with increasing n. The continued fraction then represents a function which is meromorphic over the entire finite plane. This particular result, though in a yet more restricted form, has been previously discovered by STIELTJES.* Confining his attention to a continued fraction in which a_n is real and positive, he proved that it represents a meromorphic function when $\lim. a_n = 0$, and only then.

<div align="center">§ 2.</div>

The proof of the above theorem is as follows: Let C be a circle whose center is the origin and whose radius is $e/4k$, in which e is taken to denote some fixed number less than unity. If e_1 is any other assigned number greater than e but less than unity, the value of $|a_n z|$ throughout the circle—inclusive of the perimeter—may be made less than $e_1/4$ by sufficiently increasing n. Let this be true when $n > \mu$, and consider the continued fraction

$$(2) \qquad \frac{1}{1+} \ \frac{a_{\mu+1} z}{1} \ + \ \frac{a_{\mu+2} z}{1} \ + \ \cdots .$$

We shall first demonstrate the convergence of (2). Let N_n and D_n be the numerator and denominator of its nth convergent. Obviously $2|D_2| > |D_1|$. Suppose also that

$$(3) \qquad 2|D_n| > |D_{n-1}|$$

for any particular value of n. Since

$$(4) \qquad D_{n+1} = D_n + a_{n+\mu} z D_{n-1} ,$$

it follows that

$$|D_{n+1}| > |D_n| \left(1 - \frac{e_1}{2} \right) ,$$

and hence $2|D_{n+1}| > |D_n|$. The inequality (3) therefore holds for all values of n and accordingly

$$\frac{|D_{n+1}|}{|D_{n-1}|} > \tfrac{1}{4} .$$

A similar result holds for the numerators of the convergents.

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*Annales de la Faculté des Sciences de Toulouse, vol. 9, *A* pp. 42–47.

Consider next the equation

$$
\begin{aligned}
(5) \quad \frac{N_{n+r+1}}{D_{n+r+1}} - \frac{N_n}{D_n} &= (-1)^n a_{\mu+1} a_{\mu+2} \cdots a_{\mu+n} z^n \left(\frac{1}{D_n D_{n+1}} - \frac{a_{\mu+n+1} z}{D_{n+1} D_{n+2}} \right. \\
&\quad + \frac{a_{\mu+n+1} a_{\mu+n+2} z^2}{D_{n+2} D_{n+3}} - \cdots + (-1)^r \left. \frac{a_{\mu+n+1} a_{\mu+n+2} \cdots a_{\mu+n+r} z^r}{D_{n+r} D_{n+r+1}} \right).
\end{aligned}
$$

Since $|a_{\mu+i} z| < e_1/4$ and $4|D_{i+1} D_i| > |D_i D_{i-1}|$, the modulus of the right hand member is less than

$$
\frac{e_1^n}{4^n |D_n D_{n+1}|} (1 + e_1 + e_1^2 + \cdots + e_1^r),
$$

and hence less than

$$
\frac{e_1^n}{4(1 - e_1) |D_1 D_2|}.
$$

But by increasing n the last quantity may be made as small as desired. If, therefore, an arbitrarily small quantity ϵ is prescribed, a value m can be found such that

$$
\left| \frac{N_{n+r+1}}{D_{n+r+1}} - \frac{N_n}{D_n} \right| < \epsilon \qquad\qquad (n > m).
$$

It follows that N_{n+r+1}/D_{n+r+1} converges uniformly to a limit in the circle C. The convergence of the continued fraction (2) has accordingly been established.

The character of the limiting function can next be easily determined. The inequality (3) and the similar inequality connecting N_n and N_{n-1} show, in the first place, that neither the numerator nor the denominator of the nth convergent ever vanishes. The convergents are, therefore, holomorphic throughout C. Since also they converge uniformly to a limit, their limit must be holomorphic in the interior of C. Furthermore, whenever a series of holomorphic functions converge uniformly to a limit, the zeros of the limiting function are the condensation points of the zeros of the rational functions.[*] We conclude, therefore, that the limiting function cannot vanish within C.

A case of special interest is that in which $|a_{\mu+n}| \leqq k$ for all values of n in the continued fraction (2). The radius of C may then be taken as nearly equal to $1/4k$ as desired, and we have in consequence the following result:

THEOREM II.   *If in any continued fraction*

$$
\frac{1}{1} + \frac{b_2 z}{1} + \frac{b_3 z}{1} + \cdots
$$

$|b_n|$ *is equal to or less than a fixed quantity k for all values of n, the continued fraction converges within a circle of radius $1/4k$ described about the origin of the*

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[*] HURWITZ, Mathematische Annalen, vol. 33.

z-plane as center, and it represents a function which is holomorphic within the circle and vanishes nowhere in its interior.

§ 3.

We return now to our continued fraction (1). If the mth convergent is denoted by N'_m/D'_m, we have

(6)
$$\frac{N'_{\mu+n-1}}{D'_{\mu+n-1}} = \frac{N'_{\mu-1} + a_\mu z \dfrac{N_n}{D_n} N'_{\mu-2}}{D'_{\mu-1} + a_\mu z \dfrac{N_n}{D_n} D'_{\mu-2}}.$$

But in § 2 it was shown that N_n/D_n converges uniformly to a function—call it $\phi(z)$—which is holomorphic within C. Hence the numerator and denominator of the right hand member of (6) converge uniformly, and their quotient approaches the limit

$$\frac{N'_{\mu-1} + a_\mu z N'_{\mu-2} \phi(z)}{D'_{\mu-1} + a_\mu z D'_{\mu-2} \phi(z)}.$$

It is impossible that the denominator of this expression should vanish identically, for we have

$$\frac{D'_{\mu-2}}{D'_{\mu-1}} = \frac{1}{1} + \frac{a_{\mu-1} z}{1} + \frac{a_{\mu-2} z}{1} + \cdots + \frac{a_2 z}{1},$$

which would give a second continued fraction for $\phi(z)$ inconsistent with (2).* The limit is therefore an analytic function.

Furthermore, since the numerator and denominator of the above expression are holomorphic in C, its only singularities will be poles. Let each of these poles be excluded from C by drawing a circle of small but finite radius around it. We then have left a region within which not only the numerator and denominator of the right hand member of (6) converge, but also their quotient, $N'_{\mu+n-1}/D'_{\mu+n-1}$. The convergence of the continued fraction is thus established. Since also by increasing μ the radius of C may be made as nearly equal to $1/4k$ as we please, the theorem enunciated at the beginning of the paper follows.

It will be observed that the proof involved the fact that $N'_{\mu-1}$, $N'_{\mu-2}$, $D'_{\mu-1}$, $D'_{\mu-2}$ are polynomials, and did not depend upon the monomial form of the numerators and denominators of the first $\mu - 1$ partial quotients. The theorem will therefore hold if a finite number of partial quotients are replaced by arbitrarily selected rational functions.

§ 4.

No information is afforded in theorem I concerning the character of the continued fraction without the circle considered. This question is discussed for

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* See PADE's thesis, Annales de l'Ecole Normale, sér. III, vol. 9 (1892), supplément.

an important case in the Annals of Mathematics, October, 1901, namely, for the case in which a_n approaches a finite limit k in such a manner that a_{2n-1}/a_{2n} and a_{2n}/a_{2n+1} for sufficiently great values of n can be expanded into series in ascending powers of $1/n$. It is shown that *the domain within which the function represented by the continued fraction is meromorphic can then be extended over the entire plane with the exception of a rectilinear cut from $x = -1/4k$ to $x = \infty$, drawn in a direction which is the continuation of the line from the origin to $x = -1/4k$.* Application of this result is made to the continued fraction of GAUSS.

$$\S 5.$$

Theorem II gave a sufficient condition that the limiting function shall be holomorphic within the circle in question. This condition is, however, by no means a necessary one. To obtain a more general condition that there shall be no poles of the function within the circle, consider the continued fraction

$$(7) \qquad \frac{1}{1+} \; \frac{r_2 e^{i\theta_2}}{1+} \; \frac{r_3(1-r_2)e^{i\theta_3}}{1+} \; \frac{r_4(1-r_3)e^{i\theta_4}}{1} + \cdots,$$

in which r_n is a positive number less than 1. Obviously

$$|D_2| \geqq (1-r_2)|D_1| = 1 - r_2.$$

If, now, for any value of n

$$(8) \qquad |D_n| \geqq (1-r_n)|D_{n-1}| \geqq (1-r_2)(1-r_3)\cdots(1-r_n),$$

we shall have also

$$|D_{n+1}| \equiv |D_n + r_{n+1}(1-r_n)e^{i\theta_n}D_{n-1}| \geqq |D_n| - r_{n+1}(1-r_n)|D_{n-1}|,$$

and hence

$$|D_{n+1}| \geqq |D_n|(1-r_{n+1}) \geqq (1-r_2)(1-r_3)\cdots(1-r_{n+1}).$$

The inequality (8) therefore holds for all values of n.

By a well known formula, the nth convergent is the sum of n terms of the series

$$(9) \quad 1 - \frac{r_2 e^{i\theta_2}}{D_2} + \frac{r_2 r_3(1-r_2)e^{i(\theta_2+\theta_3)}}{D_2 D_3} - \frac{r_2 r_3 r_4(1-r_2)(1-r_3)e^{i(\theta_2+\theta_3+\theta_4)}}{D_3 D_4} + \cdots.$$

In the particular case for which $\theta_n = \pi$ we have

$$D_n = (1-r_2)(1-r_3)\cdots(1-r_n),$$

and the ratio of the nth term of the series to the preceding is then $r_n/(1-r_n)$. When $r_n < 1$, this is clearly the case which is least favorable to the convergence of the continued fraction (7). The following result is now evident:

THEOREM III. *If r_n is a positive number, a necessary condition that the continued fraction*

$$\frac{1}{1+}\ \frac{r_2 e^{i\theta_2}}{1+}\ \frac{r_3(1-r_2)e^{i\theta_3}}{1}\ +\ \frac{r_4(1-r_3)e^{i\theta_4}}{1}\ +\ \cdots$$

shall converge for all values of the arguments θ_n is that the series

$$(10)\qquad 1 + \frac{r_2}{1-r_2} + \frac{r_2 r_3}{(1-r_2)(1-r_3)} + \frac{r_2 r_3 r_4}{(1-r_2)(1-r_3)(1-r_4)} + \cdots$$

shall likewise converge.   This is also a sufficient condition if $r_n < 1$ for all values of n.

§ 6.

The continued fraction

$$(11)\qquad\qquad \frac{1}{1+}\ \frac{b_2}{1+}\ \frac{b_3}{1+}\ \cdots$$

is usually capable of being expressed in the form (7).   Often we shall also have $r_n < 1$.   For example, this is true when

$$(12)\qquad\qquad |b_2| < \tfrac{1}{2},\quad |b_{2n+1}| + |b_{2n+2}| \leqq \tfrac{1}{2}\qquad\qquad (n > 0).$$

For if we put

$$|b_2| = r_2,\quad |b_n| = |1 - r_{n-1}|\, r_n\qquad\qquad (n > 2),$$

and assume that $r_{2n} < \tfrac{1}{2}$, it follows from the second of the inequalities (12) that

$$(13)\qquad\qquad r_{2n+1} < 1,\quad r_{2n+2} < \tfrac{1}{2}.$$

But $r_2 < \tfrac{1}{2}$, and hence, by induction, the inequalities (13) hold for all values of n.

PRINGSHEIM * has shown that the conditions (12) suffice to ensure the convergence of a continued fraction of the type (11).   This appears to be the most general criterion for convergence hitherto discovered except for the special case in which all the elements b_n are positive.   It is clear, however, from (13) that PRINGSHEIM's criterion is included as a special case under theorem III.

§ 7.

Assuming now r_n to be less than 1, we shall next consider the effect of multiplying a single partial numerator, $r_m(1 - r_{m-1})e^{i\theta_m}$, by a quantity δ, the absolute value of which is less than 1.   If we put $r'_m = |\delta_m|\, r_m$ and seek to preserve the form of the continued fraction unaltered, we must make

$$(1 - r_m)\, r_{m+1} = (1 - r'_m)\, r'_{m+1}.$$

Consequently $r'_{m+1} < r_{m+1}$. In like manner it follows that $r'_{m+2} < r_{m+2}$, and so on. All the terms in (10) after the $(m-1)$th will therefore be diminished, the new values being less than the old values multiplied into $|\delta|$.

Let now δ be introduced into each numerator of (7). In accordance with what has just been said, the nth convergent must be less than the sum of n terms of the series

$$1 + |\delta|\frac{r_2}{1 - r_2} + |\delta^2|\frac{r_2 r_3}{(1 - r_2)(1 - r_3)} + |\delta^3|\frac{r_2 r_3 r_4}{(1 - r_2)(1 - r_3)(1 - r_4)} + \cdots.$$

Suppose that this converges for $|\delta| = \delta'$. Then if δ in the continued fraction is replaced by a variable z, the fraction will converge uniformly in a circle having its center in the origin of the z-plane and a radius equal to δ'. Its limit is accordingly holomorphic within the circle. Furthermore,

$$|D_n| > (1 - \delta' r_2)(1 - \delta' r_3)\cdots(1 - \delta' r_n).$$

Hence it is impossible for the denominators of the convergents to vanish anywhere in the circle, and this is true also of the numerators, inasmuch as the law for their formation is similar to (4). Finally, by the theorem of HURWITZ previously cited, we conclude that the limiting function does not vanish. The result which we have thus reached may be recapitulated as

THEOREM IV. *If in the continued fraction*

$$\frac{1}{1 +}\, \frac{r_2 e^{i\theta_2}z}{1 \,+}\, \frac{r_3(1 - r_2)e^{i\theta_3}z}{1 \,+}\, \frac{r_4(1 - r_3)e^{i\theta_4}z}{1 \,+}\, \cdots$$

r_n *is a positive number less than* 1, *the continued fraction will converge uniformly in a circle described about the origin of the z-plane with a radius equal to unity, provided the series* (10) *is convergent. If this series is divergent, the radius is at least equal to the largest positive value of* δ *for which*

$$(14)\quad 1 + \delta\frac{r_2}{(1 - r_2)} + \delta^2\frac{r_2 r_3}{(1 - r_2)(1 - r_3)} + \delta^3\frac{r_2 r_3 r_4}{(1 - r_2)(1 - r_3)(1 - r_4)} + \cdots.$$

converges. The limit of the fraction is an analytic function which is holomorphic and nowhere vanishes within the circle, and the roots of the numerators and denominators of the convergents all lie without it.

An interesting application may be made to the well-known expansion

$$\tfrac{1}{2}x\log\frac{x + 1}{x - 1} = \frac{1}{1 -}\, \frac{a_2 x^{-2}}{1 \,-}\, \frac{a_3 x^{-3}}{1 \,-}\, \cdots,$$

in which

$$a_{n+1} = \frac{n \cdot n}{(2n - 1)(2n + 1)}.$$

In this case

$$\theta_n = \pi, \quad r_n = \frac{n-1}{2n-1},$$

and (14) becomes

$$1 + \tfrac{1}{2}\delta + \tfrac{1}{3}\delta^2 + \tfrac{1}{4}\delta^3 + \cdots.$$

Now this series converges when $\delta < 1$. The continued fraction therefore possesses the properties indicated in the theorem over the portion of the plane which lies without a circle of unit radius with its center in the origin. On the other hand, when $\delta = 1$, the series diverges. As the sum of n terms for $\delta = 1$ and the value of the nth convergent for $x^{-2} = 1$ are identical, the continued fraction must diverge at the points $x = \pm 1$ upon the boundary of the circle.

The properties which this continued fraction possesses exterior to the circle hold also throughout the entire imaginary domain. This can be shown by throwing it into the form

$$\frac{x}{x} - \frac{1}{b_2 x} - \frac{1}{b_3 x} - \cdots,$$

in which

$$b_{2n} = \frac{1 \cdot a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}}, \quad b_{2n+1} = \frac{a_2 a_4 \cdots a_{2n}}{a_3 a_6 \cdots a_{2n+1}},$$

Since b_n is positive and $\sum b_n$ is divergent, a theorem given on page 231 of the present volume of the Transactions can be applied. We conclude that, (1) the continued fraction converges everywhere except in the segment of the real axis between the points $x = \pm 1$, (2) the limit is holomorphic without this segment, and (3) the roots of the numerators and denominators of the convergents are contained within it. It will be noticed that the proof of this result here given, unlike previous demonstrations, is based *solely* upon the character of the continued fraction.

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